Data-Driven Portfolio Management with Quantile Constraints

Elçin Çetinkaya

Amazon.com, Seattle, WA 98144

Aurélie Thiele*

M.I.T., Cambridge, MA 02139

Abstract

We investigate an iterative, data-driven approximation to a problem where the investor seeks to maximize the expected return of her portfolio subject to a quantile constraint, given historical realizations of the stock returns. Our approach involves solving a series of linear programming problems and thus can be solved quickly for problems of large scale. We compare its performance to that of methods commonly used in the finance literature, such as fitting a Gaussian distribution to the returns ([1],[2]). We also analyze the resulting efficient frontier and extend our approach to the case where portfolio risk is measured by the inter-quartile range of its return.

Keywords: Data-driven optimization, Quantile constraints, Iterative algorithm

1. Introduction

Determining the optimal allocation of an investment budget into a portfolio of assets while achieving a trade-off between the portfolio return and the risk measure is a central task in portfolio management. Harlow [3] mentions that the most difficult and crucial task in this decision making process is defining the risk. As Cornuejols and Tütüncu [4] state, managing the risk requires a good understanding of quantitative risk measures.

Variance [5], semi-variance [6], safety-first ratio [7], Sharpe ratio [8], Sortino ratio [9], portfolio's beta [10], mean absolute deviation (MAD) [11], Gini's mean difference [12], expected regret [3], value at risk (VaR) [2], and conditional value at risk (CVaR) [13] are some of the risk measures used both in the academic

*Corresponding author

Email addresses: `elcin.cetinkaya@lehigh.edu` (Elçin Çetinkaya), `aurelie@mit.edu` (Aurélie Thiele)

literature and in practice. Stochastic dominance has also received significant interest in recent years, see for instance [14], [15]. In this paper, we focus on quantile-based risk measures such as VaR.

Benninga and Wiener [16] define VaR as "the lowest quantile of the potential losses that can occur within a given portfolio during a specific time period". In other words, it focuses on the worst anticipated loss in a pre-defined period at a given confidence level. Linsmeier and Pearson [17] motivate the need for a measure like VaR due to significant volatility in exchange rates, interest rates, and commodity prices, in addition to the increasing popularity of derivative instruments. Kim and Powell [18] argue that the quantile function is a relatively reliable measure of risk adjusted return even in a very volatile environment. The authors add that quantile optimization is robust and a reasonable procedure in a complex, volatile, and heavy-tailed environment. In addition, VaR has been widely used since JP Morgan's endeavor to standardize risk measurement throughout the market in 1994. Later, the Basel Capital Accords of 1996 let banks calculate their capital requirements for market risk according to their own VaR models, and the U.S. Securities and Exchange Commission suggested VaR as one of the three possible disclosure methods in 1997 [17].

Rodriguez [19] uses the portfolio optimization problem with VaR constraint as an example of stochastic programming with chance constraint. Solution techniques of these problems involve non-gradient-based or gradient-based techniques. In addition, nonlinear optimization techniques, where Monte Carlo simulation procedures are applied to determine the gradients of the probability functions, are also mentioned in the literature. Uryasev [20] presents developments in probabilistic constrained optimization up to 2000. Naumov and Kibzun [21] present an approach to optimize the unconditional quantile function with a General Minimax Approach (GMA), which provides an upper bound on the optimal value of the objective function. Pankov, Platonov and Semenikhin [22] investigate a single-step portfolio management problem with a quantile criterion. The authors provide a modeling and solution of a minimax optimization problem with a quantile criterion. Moreover, El-Ghaoui, Oks and Oustry [23] assume that only the bounds of the mean and the covariance matrix of the return distribution are known, and formulate the worst-case VaR optimization problem as a semi-definite programming problem. Benati and Rizzi [24] formulate the portfolio optimization problem with a VaR criterion via a data-driven mixed integer linear programming. The model is NP-hard. However, a polynomial time algorithm exists when the data set is not large. Gaivoronski and Pflug [25] provide a computational approach for VaR portfolio optimization that approximates historical VaR using smoothed VaR, which filters out local irregularities, and show that the efficient frontier differs quite significantly from that obtained in a mean-CVaR framework. Fabozzi et. al. [26] discuss the state of the art in robust portfolio management up to 2007, including quantile models.

More recently, Kim and Powell [18] develop a provably convergent algorithm that optimizes the quantile of a random function in a heavy-tailed environment. The suggested algorithm replaces the stochastic gradient with the asymmetric signum function. Wozabal [27] formulates VaR as the difference between two CVaR. The author solves the portfolio optimization problem with a VaR constraint by the difference of convex (DCA) algorithm. Goh et. al. [28] investigate VaR optimization for asymmetrically distributed returns by partitioning asset distributions between positive and negative half-spaces and minimizing a new measure called Partitioned Value-at-Risk. Zymler, Kuhn and Rustem [29] provide tractable, conservative approximations to the worst-case VaR of a derivative portfolio using the delta-gamma approximation. An extensive treatment of risk and uncertainty measures in the context of portfolio optimization is also available in Rachev, Stoyanov and Fabozzi [30]. Beasley [31] provides a tutorial on models and solution approaches for portfolio optimization up to 2013.

In this paper, we propose a fast-convergent approximation method for the portfolio management problem with quantile a criterion, which is tractable and leads to a close-to-optimal solution in numerical studies. It involves solving a series of linear problems iteratively and, therefore, can solve problems of large scale quickly. We extend the proposed method to the inter-quartile risk management problem where the risk, which is measured as the difference between two specified quartile levels, is minimized while the expected portfolio return is assured to be no worse than a specified level for the given data set.

We set up the problems in Section 2. Solution approaches are provided in Section 3. Section 4 contains the numerical experiments.

2. Problem Setup

2.1. Portfolio Management with Quantile Constraints

We aim to maximize the expected return of a portfolio of stocks, or more generally a random objective bilinear in the decision variables and the random variables, while guaranteeing that the random objective achieves a target with a given probability based on a finite set of historical scenarios. We will use the following notation:

- n : the number of decision variables, i.e., assets,
- x_i : the dollar amount in asset i,
- W : the current wealth i,
- μ : the sample mean vector of the returns,
- Q : the covariance matrix of the returns,
- τ : the target expected portfolio return,
- T : the number of observations, e.g., time periods in historical data set,
- r_{ti} : the t-th observation of random variable i, e.g., return of stock i on day t,
- α : the specified quantile level, $\alpha \in (0, 1)$,
- m : the index of the observation corresponding to the $100\alpha^{th}$ quantile., $m = \lceil \alpha \cdot T \rceil$,
- q_m : the desired value for the $100\alpha^{th}$-quantile (m-th smallest observation),
- $y_{(k)}$: the k-th smallest value in the set $(y_1, \ldots, y_n)$ for $k = 1, \ldots, n$,
- z_t : binary variable, equal to 1 when the portfolio return in that scenario t is among the $m-1$ smallest ones,
- X : the feasible set for the asset allocation, for instance including sector limits, limits in the amount of asset i sold, bought or held.

The portfolio management problem where the manager seeks to maximize expected return with a quantile constraint guaranteeing the portfolio return will be at least q_m with a given probability level α, can be formulated as:

$$\begin{aligned} \max \quad & \frac{1}{W} \sum_{i=1}^{n} \mu_i \, x_i \\ \text{s.t.} \quad & \frac{1}{W} \left(\sum_{i=1}^{n} r_{\cdot i} \, x_i \right)_{(m)} \geq q_m, \\ & x \in X, \end{aligned} \tag{1}$$

where $(1/W) \left(\sum_{i=1}^{n} r_{\cdot i} \, x_i \right)_{(m)}$ refers to the m-th lowest value of the portfolio return, with m the observation rank that corresponds to the confidence level α, as explained in the notations.

If $m = 1$, Problem (1) can be linearized easily, because the portfolio return must then be at least the threshold in *all* realizations. However, if $m > 1$, Problem (1) is hard to solve because it involves ranking the objective (portfolio return) values for every candidate solution. Our goal is to investigate an efficient approximation approach to solve Problem (1) for the case when $m > 1$, using binary variables to identify the scenarios for which the portfolio return will be at least the threshold.

2.2. Extension to Interquartile Range Minimization

Our approximation approach can be extended to any risk management problem that uses quantiles of the portfolio return to define risk, in particular, it also applies to problems where risk is defined as the inter-quantile range (IQR) of a random variable, i.e., the 75^{th} percentile minus the 25^{th} percentile of a variable such as a portfolio return, or more generally, to any difference of quantiles. The IQR measure is commonly used in financial management to quantify risk but has not been used so far in the context of portfolio optimization due to the difficulty in optimizing quantiles. To the best of our knowledge, we are the first to explicitly provide a tractable approximation for IQR minimization in portfolio management.

In what follows, we will refer to a, respectively b, as the rank of the observation corresponding to the lower, respectively higher, quantile considered. The problem, using notation similar to Problem (1), can be formulated as:

$$\begin{aligned} \min \quad & \left(\sum_{i=1}^{n} r_{t,i} x_i \right)_{(b)} - \left(\sum_{i=1}^{n} r_{t,i} x_i \right)_{(a)} \\ \text{s.t.} \quad & \sum_{i=1}^{n} \mu_i x_i \geq \tau, \\ & x \in X. \end{aligned} \tag{2}$$

Our paper focuses on using observation ranks in the data set (such as m, a, b), or more specifically, whether or not a scenario is of rank *at most* $m-1$, $a-1$ or b, respectively, in order to aid the decision-maker obtain high-quality approximate solutions to the quantile management problem using an iterative approximate algorithm.

3. Solution approach

From a methodological (solution method) perspective, the key contribution of our paper is to provide a heuristic method to the portfolio optimization problem where the decision-maker identifies the historical returns corresponding to the worst $m-1$, $a-1$ and b portfolio return values using appropriately chosen binary variables. This is made more precise below.

3.1. Portfolio optimization with quantile constraints

Considering Problem (1), we approach constraint $\frac{1}{W}\left(\sum_{i=1}^{n} r_{\cdot i}\, x_i\right)_{(m)} \geq q_m$ by determining heuristically for which of the T scenarios under consideration the portfolio return constraint should be satisfied, i.e., the portfolio return should be at least q_m. There should be at least $T-m+1$ scenarios for which the portfolio return is at least equal to the m-th smallest portfolio return, and $m-1$ that are no higher than this threshold value.

This insight allows us to explain our algorithm as follows. The specific method we analyze involves solving a linear problem iteratively where the set of constraints used to express the quantile constraint (the $m-1$ scenarios mentioned above) changes at each iteration. At each iteration, a set of portfolio return scenarios for which the objective must equal or exceed the threshold is determined. Next, the linear problem maximizes the expected portfolio return with the constraints corresponding to these scenarios. Specifically, $m-1$ scenarios for which the objective does *not* need to exceed the threshold are identified by ranking the portfolio return scenarios in ascending order, identifying the first $m-1$ of them by the vector $z \in R^T$.

When $z_t = 1$ for a scenario t, the portfolio return calculated in that scenario t does not need to exceed the threshold level q_m since only the m^{th} greatest portfolio return or higher should attain or exceed the threshold q_m. The master problem to obtain the decision allocation is then formulated as:

$$\begin{aligned} \max \quad & \frac{1}{W}\sum_{i=1}^{n} \mu_i x_i \\ \text{s.t.} \quad & (1-z_t)\frac{1}{W}\sum_{i=1}^{n} r_{t,i} x_i \geq q_m(1-z_t), \;\; \forall t \in \{1,\ldots,T\}, \\ & x \in X. \end{aligned} \tag{3}$$

We repeat solving Problem (3) for a scenario identification vector z and then update the vector z based on the latest portfolio allocation decision x iteratively until the algorithm stops to an approximate solution of the original problem. We provide our heuristic in more details below.

Algorithm 3.1.

Step 1 *Start with a feasible solution $x \in X$ to serve as a candidate solution $\bar{x}$ and set the iteration number, $s = 1$.*

Step 2 *Obtain a new active scenario selection decision z, namely z^s for the candidate solution $\bar{x}$. If there are more than one scenarios leading to the same portfolio return value and they are both candidates to be the $(m-1)^{th}$ scenario based on the current investment decision, then select the one with the smallest index.*

Step 3 *Solve the linear problem for z^s identified in Step 2. Obtain a new candidate solution, x^s, and set $s = s + 1$ and $\bar{x} = x^s$.*

Step 4 *Repeat Steps 2 and 3 until the algorithm generates the same set of active scenarios or the same candidate solution $x \in X$ in two consecutive steps, whichever happens sooner.*

We make the following additional comments on our heuristic. First, note that the feasible region of the master problem is a closed polyhedron, therefore in the case of multiple solutions at, an interior point method algorithm terminates at the analytic center of the optimal face (see Colombo [32]). Namely, if the master problem is solved by an interior point algorithm, the algorithm will have a unique solution at each iteration s for a given passive scenario set $A^s\{i : 1 \leq i \leq T \cap z_i^s = 1\}$. The algorithm will terminate if the sets A^s and A^{s+1} are identical, since the constraint sets for the Problem (3) will be identical for the iterations s and $s + 1$ which lead to the same solution under the assumption that it is solved by interior point method.

Secondly, if there is one scenario, scenario-t, belonging to set A^{s+1} but not the set A^s, then the portfolio return value obtained by scenario-t is less than or equal to that obtained by the scenario-t' which is in the set A^s but not in the set A^{s+1}. If these two scenarios lead to the same portfolio return value for the given investment decision x^s, the current investment decision x_s will be a feasible decision with the set of constraints $r_j'x \geq q_m$, $\forall\, j \in A^{s+1}$, therefore the objective function value at iteration $s + 1$ will improve with a new investment decision or stay the same with the same investment decision and the algorithm will terminate. If the scenario $t \in A^{s+1}$ leads to a lower portfolio return value than the scenario-t' in A^s for the decision x^s, then this implies that the scenario-t' in set A^s leads to a higher portfolio return value than the target q_m, since at iteration s the master problem is solved while $z_t = 0$ and the solution x^s satisfies the inequality $r_t'x^s \geq qm$. Considering that the portfolio return obtained by the scenario-t' is greater than the portfolio return value obtained by the scenario-t' which is greater than or equal to q_m, we can conclude that the quantile target is satisfied for a smaller probability level at iteration s. Therefore, transforming the active set from A^s to A^{s+1} enlarges the feasible region for the master problem and it provides an improved objective function value. For the cases where there are more than one scenarios belonging to the set A^{s+1} but not to the set A^s, the same argument is also valid. Therefore, the proposed algorithm improves at each iteration until it stops.

3.2. Minimization of interquartile range

The interest of the IQR extension lies in the use of two quantile constraints. Specifically, Problem (2) can be written as:

$$\begin{aligned} \min \quad & q_b - q_a \\ \text{s.t.} \quad & q_a \leq \left(\sum_{i=1}^{n} r_{t,i} x_i \right)_{(a)} \\ & q_b \geq \left(\sum_{i=1}^{n} r_{t,i} x_i \right)_{(b)} \\ & \sum_{i=1}^{n} \mu_i x_i \geq \tau, \\ & x \in X. \end{aligned}$$

We define two auxiliary problems. The first one identifies b scenarios for which the portfolio return must not exceed q_b. Therefore, it actually determines scenarios leading b smallest portfolio return values while investment decision is given. The second one detects $a - T + 1$ scenarios for which the portfolio return must be at least q_a, or equivalently, the scenarios corresponding to the $a - 1$ smallest return values for which the constraints do not need to be enforced.

For specific ranks a and b, the vectors z^a and z^b are the solutions of the following auxiliary problems:

$$\begin{aligned} \min \quad & \sum_{t=1}^{T} \left(\sum_{i=1}^{n} x_i r_{t,i} \right) z_t^a \\ \text{s.t.} \quad & \sum_{t=1}^{T} z_t^a = a - 1, \\ & 0 \leq z_t^a \leq 1, \ \forall t. \end{aligned} \tag{4}$$

and

$$\begin{aligned} \min \quad & \sum_{t=1}^{T} \left(\sum_{i=1}^{n} x_i r_{t,i} \right) z_t^b \\ \text{s.t.} \quad & \sum_{t=1}^{T} z_t^b = b, \\ & 0 \leq z_t^b \leq 1, \ \forall t. \end{aligned} \tag{5}$$

We need to rank the scenarios in order to determine the worst $a-1$ and b scenarios while the investment decision is given. For specific ranks a and b, the vectors z^a and z^b are the active-scenario identification vectors such that we have $z_t^a = 1$, if scenario t is among those that achieve the $a-1$ smallest returns and $z_t^b = 1$, if scenario t among those that achieve b smallest returns. The vectors z^a and z^b will have $a-1+T-b$ values in common ($a-1$ "ones" and $T-b$ "zeros").

The problem to obtain a decision allocation for given ranking vectors z_t^a and z_t^b is formulated as follows:

$$\begin{aligned}
\min_{q_m, q_a, x} \quad & q_b - q_a \\
\text{s.t.} \quad & (1 - z_t^a) q_a \leq \sum_{i=1}^{n} r_{t,i} x_i (1 - z_t^a), \ \forall t, \\
& z_t^b q_b \geq \sum_{i=1}^{n} x_i r_{t,i} z_t^b, \ \forall t, \\
& \sum_{i=1}^{n} \mu_i x_i \geq \tau, \\
& x \in X.
\end{aligned} \tag{6}$$

The algorithm for the interquantile range management problem is very similar to Algorithm 3.1. The only difference is that we need to identify both $a-1$ and b worst-case scenarios instead of $m-1$ scenarios as in Algorithm 3.1.

Algorithm 3.2.

Step 1 *Start with a feasible solution $x \in X$ to serve as a candidate solution $\bar{x}$ and set iteration number, $s = 1$.*

Step 2 *Solve Problem 4 and Problem 5 for the candidate solution $\bar{x}$, and obtain a new z^a and z^b, namely $z^{a,s}$ and $z^{b,s}$.*

Step 3 *Solve the master problem for $z^{a,s}$ and $z^{b,s}$ identified in Step 2. Obtain a new candidate solution, x^s, and set $s = s+1$ and $\bar{x} = x^s$.*

Step 4 *Repeat Steps 2 and 3 until the algorithm generates the same corner point of for Problem 4 and Problem 5 or the same candidate solution $x \in X$ in two consecutive steps, whichever happens sooner.*

4. Numerical Results

From an algorithmic perspective, the contribution of our paper is to test the performance our rather naive iterative algorithm using historical data and compare it with that of other, better-known benchmarks. Our results suggest that, although our algorithm is a heuristic, this simple technique deserves becoming better known among practitioners due to its strong empirical performance in terms of solution time and solution quality.

4.1. Portfolio optimization with quantile constraint

This section tests the performance of Algorithm 3.1 in terms of solution time and solution quality. As comparison benchmarks, we use portfolio optimization models with quantile constraints where the asset returns are assumed to be Normally and Log-Normally distributed as in [1], [33], and [2]. We refer to these benchmark models as the "Normal Approximation" and the "Log-Normal Approximation" methods. In particular, we compare our solution with the optimal solution in the Normal Approximation method which assumes that the portfolio return is a Normally distributed random variable and that in the Log-Normal Approximation method which is built by approximating the portfolio return by a Log-Normally distributed random variable based on a moment matching approach. The motivation for this choice is that decision-makers might make the Normal or Log-Normal assumption due to the increased tractability, even when they know the true returns do not obey these distributions. In addition, a data-driven iterative VaR optimization algorithm introduced by Larsen, Mausser, and Uryasev [34] (Algorithm-A1 thereafter) is used an another benchmark model. Algorithm A1 provides an approximated solution to the quantile optimization problem by iteratively solving a linear optimization problem which maximizes the CVaR of the portfolio return and was first introduced by Rockafellar and Uryasev [2].

Our results suggest that:

1. Our iterative algorithm converges in a small number of iterations. Total solution time in terms of CPU seconds is close to that obtained with the benchmark methods. Indeed, in some experiments with relatively small observations, the Linear Approximation methods terminates earlier than the benchmark models, especially the Log-Normal Approximation method.
2. For a given data set, the number iterations and time to reach a solution change for the proposed Linear Approximation method in a set of experiments, however those for Algorithm-A1 stay relatively consistent within the same data set. The upper bounds of the ranges of the observed number of iterations and time to convergence for the Linear Approximation problem in different numerical experiments are closer to the number of iterations and time to convergence for Algorithm-A1 than the lower bounds of the ranges.
3. The proposed method generally outperforms the benchmark methods in terms of return-risk efficiency in both in-sample and out-of-sample performance tests. That is, for a given quantile target q_m, the proposed algorithm generally leads to a portfolio allocation decision providing higher expected portfolio return with both the training and testing data set.
4. Portfolios generated by the proposed algorithm are generally more robust against unexpected stock return realizations in the out-of-sample data sets than the ones generated by the benchmark methods.

4.1.1. Setup

A traditional approach for the portfolio management problem with quantile constraints assumes that the asset returns follow a jointly Gaussian distribution because this special case can be formulated as a (more tractable) second-order cone problem. In other words, the quantile constraint, which is hard to formulate, is in general approximated by the quantile function of a Normal distribution. Another approach, which is known as the Fenton-Wilkinson Method [35], calculates an approximation to the Log-Normal sum distribution based on a moment matching method. In contrast with the Gaussian case, a linear combination of Log-Normal random variables is not Log-Normal, therefore this is an approximation even if each single stock return series obeys a Log-Normal distribution. The Fenton-Wilkinson method approximates the Log-Normal sum by a single Log-Normal random variable by matching the first and the second moments. Therefore, we will refer to these models as the Normal and the Log-Normal Approximation methods, respectively. Our proposed algorithm will be referred as the Linear Approximation method since it involves solving a series of linear problems.

The portfolio management problem according to the Normal Approximation method for a given α probability level is formulated as:

$$\begin{array}{ll} \max & \frac{1}{W}\mu^T x \\ \text{s.t.} & \mu^T x + \phi^{-1}(\alpha)\sqrt{x^T Q x} \geq q_m W, \\ & x \in X, \end{array} \tag{7}$$

where ϕ is the CDF of a standard Gaussian random variable.

The portfolio management problem according to the Log-Normal Approximation (Fenton-Wilkinson) method for a given α probability level is written as follows:

$$\begin{array}{ll} \max & \frac{1}{W}b^T x \\ \text{s.t.} & 2\ln(b^T x) - \dfrac{1}{2}\ln(x^T A x) + \phi^{-1}(\alpha)\sqrt{\ln(b^T x) - 2\ln(x^T A x)} \geq \ln(W q_m), \\ & x \in X, \end{array} \tag{8}$$

where the vector $b \in \mathcal{R}^n$ is such that

$$b_i = e^{\left(\tilde{\mu_i} T + \frac{\tilde{\sigma_i}^2 T}{2}\right)} \; \forall i,$$

and the matrix A $\in \mathcal{R}^{nxn}$ is such that

$$A_{i,j} = e^{\left((\tilde{\mu_i} + \tilde{\mu_j})T + \frac{T}{2}(\tilde{\sigma_i}^2 + \tilde{\sigma_j}^2 + 2\rho_{i,j}\tilde{\sigma_i}\tilde{\sigma_j})\right)} \; \forall i, \; \forall j \text{ and } i \neq j$$

$$A_{i,i} = e^{2T\tilde{\mu_i} + 2T\tilde{\sigma_i}^2} \; \forall i.$$

The explanation of the Fenton-Wilkinson method and derivation of the Log-Normal approximation problem are provided in Appendix 1.

Note that the objective function formulation according to this approach differs from that of the Linear Approximation method. In order to have a fair comparison, we update this benchmark model so that the objective function is the same as that of the Linear Approximation method (namely, the sample average of return rates), while the quantile function is approximated according to the Fenton-Wilkinson method. This hybrid benchmark model is:

$$\begin{aligned} \max \quad & \tfrac{1}{W}\mu^T x \\ \text{s.t.} \quad & 2\ln(b^T x) - \frac{1}{2}\ln(x^T A x) + \phi^{-1}(\alpha)\sqrt{\ln(b^T x) - 2\ln(x^T A x)} \geq \ln(W q_m), \\ & x \in X. \end{aligned} \tag{9}$$

Algorithm-A1 provides an approximated solution to the quantile maximization problem by iteratively maximizing the tail conditional expectation of the portfolio return for updated probability levels so that at the next iteration the new tail conditional expectation, which will be maximized (by using the linear problem suggested by Rockafellar and Uryasev [2]) is a closer lower bound to the original quantile level of interest. The linear tail conditional expectation optimization problem and the algorithm introduced by Larsen, Mausser, and Uryasev [34] is adjusted to our problem setting as follows:

Algorithm 4.1.

Step 1 *Assign a lower bound on the expected portfolio return, the probability level parameter for the tail conditional expectation, and a value for the algorithm constant ζ, $0 \leq \zeta \leq 1$.*

Step 2 *Set $\alpha_0 = \alpha$ and $s = 0$.*

Step 3 *Solve the tail conditional expectation maximization problem:*

$$\begin{aligned} \max_{x^+, x^-, x, \kappa} \quad & \frac{1}{W t_s} \sum_{t=1}^{t_s} \left(\sum_{i=1}^{n} r_{.i} x_i \right)_t \\ s.t. \quad & \mu^T x \geq \eta, \\ & \sum_{i=1}^{n} r_{t,i} x_i \geq \kappa \ \forall t \geq t_s, \\ & \sum_{i=1}^{n} r_{t,i} x_i \leq \kappa \ \forall t < t_s, \\ & x \in X. \end{aligned} \tag{10}$$

Step 4 *Sort the scenarios according to their return values $\frac{1}{W}\sum_{i=1}^{n} r_{t,i} x_i^s$ based on the solution of the Problem (10) at iteration s.*

Step 5 *Set $s = s+1$, $b_s = \alpha + (1-\alpha)(1-\zeta)^s$, $t_s = \lfloor T(1-b_s) \rfloor$, and $\alpha_s = 1 - \frac{1-\alpha}{b_s}$.*

Step 6 *If $t_s \leq \lfloor \lceil T\alpha \rceil \rfloor$ repeat Step 3,4, and 5, otherwise exit.*

In the numerical experiments below, the constant ζ is set to 0.5.

4.1.2. Time and Number of Iterations to Convergence

We compare the CPU seconds used by the solver calls (variable *_solve_time*) for each approach using different quantile targets over different data sets with varying sample sizes and number of assets. The Mosek solver is used through the AMPL modeling language on a 2.10 GHz Pentium(R) machine. The results are provided in Table 1. The number of decision variables increases with the number of stocks considered in all the approximation methods. However, as the number of scenario increases, the number of constraints of the Linear Approximation method and Algorithm-A1 increases. Therefore, the total time spent by solvers for these methods is more vulnerable to the data set size than those in the Normal and LogNormal Approximation methods. In addition, generally the Linear Approximation method requires fewer iterations and less time to terminate than Algorithm-A1.

Table 1: Total Solution Time in CPU Seconds

Data Set			Linear Approx.		Normal Approx.		LogNormal Approx.		Algorithm. A1		
Sample Size	Asset Number	Iteration Range	Min. Solution Time	Max. Solution Time	Min. Solution Time	Max. Solution Time	Min. Solution Time	Max. Solution Time	Iteration Range	Min. Solution Time	Max. Solution Time
100	30	[2,4]	0.0920	0.1440	0.0960	0.1480	0.1680	0.2000	4	0.1480	0.1760
1000	30	[4,9]	0.4920	1.4081	0.0960	0.1600	0.1520	0.1760	7	1.1041	1.1481
2000	30	[4,15]	1.0161	5.1283	0.0800	0.1480	0.1400	0.1640	8	2.5242	2.6962
5000	30	[3,15]	0.7000	5.0763	0.0960	0.1560	0.1400	0.1680	8	2.5242	2.6642
100	50	[2,4]	0.1480	0.2440	0.1880	0.5080	0.7120	0.7961	4	0.2680	0.2920
1000	50	[2,11]	0.9721	3.4482	0.2120	0.5400	0.8201	0.8441	7	1.9881	2.0681
2000	50	[4,11]	2.0521	7.1324	0.4640	0.5320	0.7561	0.8481	8	4.9883	5.2763
5000	50	[3,11]	5.7444	20.3210	0.4600	0.5080	0.7320	0.8521	9	15.5410	16.5650
100	100	[3,5]	0.4720	1.0481	1.4161	3.2122	9.7846	11.2530	[4,5]	0.8081	1.4161
1000	100	[2,9]	2.7602	8.0485	2.4322	3.1122	11.2050	12.7370	7	5.2363	5.3403
2000	100	[5,11]	7.7685	21.0010	2.5482	3.2962	10.9010	12.9330	8	12.4770	13.9930
5000	100	[4,8]	16.4010	38.2580	2.8202	3.4042	9.7006	12.3130	9	40.6790	43.7510
100	200	[3,15]	0.6880	5.1283	0.1000	0.1520	0.1480	0.1640	[4,5]	1.1921	1.6401
1000	200	[5,11]	54.2030	116.0000	12.0210	13.2890	101.2400	135.9100	[7,9]	63.5200	88.9060
2000	200	[6,11]	25.6100	56.3640	9.8606	14.2730	98.4500	104.8500	8	31.8100	34.5500
5000	200	[4,11]	40.6490	253.1600	2.8802	21.6890	74.2130	116.6800	9	69.8720	132.3400

Each row in Table 1 summarizes a set of experiments conducted with different quantile targets (between 0.90 and 1.09) over the same training data identified by the number of assets and sample size. For each set of experiments, the minimum and maximum values of the observed solution time values for each approximation method are recorded in CPU seconds. In addition, the minimum and maximum values of the observed number of iterations to converge for the iterative methods are also presented.

4.1.3. Performance of approximation methods

We now analyze the performance of approximation methods based on two different types of data sets, namely training and testing data sets. The allocation decision is determined based on the training data set for each approach (Linear, Normal, and Log-Normal Approximation methods, and Algorithm-A1). The Linear, Normal, and Log-Normal Approximation methods are run for given quantile targets to obtain the highest expected portfolio return. Then, Algorithm-A1 is run for each expected return-quantile target pair of the Linear Approximation method to maximize the approximated quantile function.

Furthermore, we compare the portfolio return rates based on these allocation decisions over the testing data sets, which are used as out-of-sample data sets for testing purposes. In addition, we use a performance measure (ω) which is the ratio between the portfolio return realization of the Linear Approximation method and that of the benchmark method with the training and testing data set. We provide 95% one-sided confidence intervals (CI) of ω for each case.

We follow the same approach for four different numerical experiments sets. In each of these sets, a different scenario generation method is used. Also, in each set of numerical experiments three different time-period lengths (daily, weekly, and monthly stock return scenarios) are considered. The stocks the manager can invest in are 30, 50, or 100 stocks listed on the New York Stock Exchange (NYSE) in all of the four sets of numerical experiments.

The goal of the proposed approach is to manage the downside risk. Therefore, we are particularly interested in the low quantile values such as the 5^{th} percentile. The quantile constraints will enforce that approximated quantile function values do not fall below the pre-specified q_m level 95% of the time (hence $\phi^{-1}(\alpha) = 1.645$). The set of feasible allocation X incorporates transaction costs constraints, with unit transaction coefficients assumed to be 0.02. The lower and upper bounds on the holding in a single asset are selected to be 0% and 30% of the overall wealth. Limit on sector holdings, also incorporated in the definition of X, is assumed to be 50%. Simulations were performed using MATLAB R2012a and R Statistical Software. The reader is referred to Pfaff [36] for an introduction to financial modeling using R.

Numerical Experiments Set 1

In this set of numerical experiments, three training data sets are composed of 100 daily, weekly, and monthly rate of return (ROR) observations of 30, 50, and 100 stocks listed in the New York Stock Exchange (NYSE). For each case, a testing data set is a random data set of 100 scenarios generated by Monte-Carlo simulation assuming that the stock returns follow a multivariate t-distribution. The motivation for this distribution choice is that it has fat tails and thus can generate more adverse events than have been observed in the historical data set. This allows us to gain insights into the robustness of the solution obtained by our algorithm to left-tail risk and adverse events.

The parameters of the multivariate distribution are extracted from the historical data. In addition, for each stock a set of additional noisy data is generated from its left tail distribution (5^{th} percentile and lower). The additional noisy data are included in each testing data set in order to ensure that the empirical probability distribution of the return rate has heavier left tail. In other words, we seek to compare the dependence between random stock returns, and the fat tailed nature of stock returns by the t-copula and additional adverse return realizations. This way, we compare the robustness of the approximation methods against unexpected return rate movements within a similar (perturbed by additional noisy data) interdependence structure of the stock market.

In other words, the losses (rate of return values less than 1) are more likely to occur in testing data sets than in corresponding training data sets. This lets us compare the robustness of the approximation methods against undesired realizations of the stock returns. In other words, if the Linear Approximation approach performs better than benchmark models in terms of return-risk efficiency, then it can be inferred that the Linear Approximation method is more robust against undesired return movements within a similar (perturbed by additional noisy data) interdependence structure of the stock market.

The portfolio return realization according to both the testing (out-of-sample) and training (in-sample) data sets are calculated based on the allocation decision obtained over the training data for all of the approximation methods. Next, 95% confidence intervals for ω are calculated in order to compare the risk-return performance of the linear approximation method with that of the benchmark models.

Table 2 provides the 95% CI of ω for all benchmark models over both the testing and training data sets.The relative performance of the Linear Approximation method with respect to the Algorithm-A1 over the training data set is not provided, since the expected portfolio return values of these approaches over the training data sets are the same. Table 2 suggests that generally Linear Approximation method's investment decisions perform better than those of the benchmark models in both the testing and training data sets (except Algorithm-A1) with more than 95% confidence.

Numerical Experiments Set 2

Here, we test the performance of the Linear Approximation method while daily, weekly and monthly training data sets are generated according to Monte Carlo simulation with Geometric Brownian Motion (GBM) and corresponding testing data sets are historical stock return observations. That is, 100 daily, weekly and monthly historical observations of 30, 50 and 100 stocks (listed on NYSE) are used to forecast stock return realizations for the following 100 days, 100 weeks and 100 months, respectively. These daily, weekly and, monthly stock return forecasts are used as training data sets and the actual stock return values during the same period are used as testing data sets. Investment decisions according to all of the quantile management approaches are determined based on these training data sets. Portfolio return realizations of these investment decisions with the actual stock returns (testing data) are compared.

95% one-sided confidence intervals (CI) of ω for each case are constructed in order to compare the portfolio return realizations with both the testing and training data sets. Table 2 summarizes the results. They suggest that the Linear Approximation method and the benchmark models' performances are similar when the data frequency is a day. However, the Normal Approximation benchmark method provides better portfolio return realizations in testing data for some observations. On the other hand, the Linear Approximation method outperforms the benchmark methods with 95% confidence when the data frequency is a month and a week in both testing and training data sets.

Numerical Experiments Set 3
Here, we generate three training data sets (daily, weekly, and monthly) using the Fama-French three-factor model. Daily, weekly and monthly series of factors are obtained from Prof. Kenneth R. French's website at: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/. That is, 100 daily, 100 weekly, and 100 monthly historical factor values are used to construct the Fama-French three-factor model for each of the 30, 50, and 100 NYSE stocks considered. Next, stock returns of each stock for the following 100 days, 100 weeks, and 100 months respectively are forecast according to the corresponding three-factor model. The actual stock returns during the same periods are used as testing data sets. Investment decisions according to each method are determined based on training data sets. Portfolio return realizations over both actual stock return observations (testing data) and training data are compared. According to Table 2, portfolios generated by the Linear Approximation method usually lead to higher return values than those generated by benchmark methods with both training and testing data sets.

Numerical Experiments Set 4
Here, we generate daily, weekly and monthly training data sets by using a multi-factor model with macro factors. We follow the forecasting approach presented in a working paper of the International Monetary Fund (Oyama [37]). First, effective macro factors are selected among 10 macro factors by principal component analysis (PCA) using 100 daily, weekly and monthly observations of each factor. The macro factors are West Texas Intermediate (WTI) Crude Oil Spot Price, Dow Jones Industrial Average Index (DJI), Aruoba-Diebold-Scotti (ADS) Business Conditions Index, US Dollar to Japanese Yen Exchange Rate, EURO to US Dollar Exchange Rate, Chicago Board of Options Exchange (CBOE) Volatility Index (VIX), BofA Merrill Lynch US Corp AA Total Return Index, BofA Merrill Lynch US Corp BBB Total Return Index, 1-Year Treasury Constant Maturity Rate and 3-Month Treasury Constant Maturity Rate. Regarding our 30, 50 and 100 stocks, we obtain each stock's exposure to macro factors by regressing its returns on the series of daily, weekly and monthly changes in growth rates of the macro factors over the estimation period.

During the principal component analysis, both the Kaiser criterion and the value of the cumulative proportion of variance explained by the components are considered. That is, the components whose corresponding eigenvalues are greater than 1 are accepted. If the cumulative proportion of variance explained by the components is less than 80%, an additional component with the next highest eigenvalue is accepted as well. Effective factors are selected by associating each component with a factor by the VARIMAX rotation method in Principal Component Analysis (PCA). In our study, when the data frequency is a day, a week, and a month, the number of effective macro factors are five, five, and four respectively.

Oyama [37] uses the residual of each regression model as an index representing the information explained by the market but not by other variables and uses this index as another factor. We follow the same approach and regress each individual stock's returns on effective macro factors and on this residual index in order to obtain the factor loadings for each stock. We treat each factor as a stationary time series (according to Augmented Dickey-Fuller (ADF) test results) and fit a suitable Autoregressive Moving Average (ARMA) model to it by considering the autocorrelation function, the partial autocorrelation function and the maximum likelihood function value. In addition, the quality of fit for each time series is controlled via residual analysis. We generate 100 daily, weekly, and monthly future scenarios for each factor based on its corresponding time series model. Next, future return scenarios for each individual stock are calculated according to the corresponding multi-factor model. These forecast scenarios stand for the training data set and investment decisions are made based on this training data set. Actual stock return realizations over the same period form the testing data set.

According to Table 2, when the data frequency is a month the Normal Approximation method provides higher expected portfolio return values than the Linear Approximation method over testing data sets for given quantile targets.

Table 2: Relative Risk-Return Efficiency of the Linear Approximation Method with respect to the Benchmark Models

			– Benchmark:	Normal	Approx.		–Benchmark:	LogNormal	Approx.		–Benchmark:	Algorithm	A1
Data Set	Data Freq.	Asset Size	Testing Set Mean	Testing Set L.Bound	Training Set Mean	Training Set L.Bound	Testing Set Mean	Testing Set L.Bound	Training Set Mean	Training Set L.Bound	Testing Data Mean	Testing Data L.Bound	
Set-1	Day	30	1.00081	1.00065	1.00200	1.00133	1.00794	1.00776	1.00741	1.00723	1.00013	1.00012	
Set-1	Week	30	1.00136	1.00072	1.00633	1.00616	1.01205	1.01184	1.01156	1.01135	1.00012	1.00011	
Set-1	Month	30	1.01456	1.00899	1.01108	1.01083	1.02333	1.02285	1.02357	1.02261	1.00005	1.00002	
Set-1	Day	50	1.00094	1.00029	1.00075	1.00021	1.00830	1.00814	1.00746	1.00730	1.00011	1.00011	
Set-1	Week	50	1.00098	1.00039	1.00367	1.00361	1.00883	1.00870	1.00922	1.00830	1.00012	1.00011	
Set-1	Month	50	1.02058	1.01118	1.01113	1.01098	1.01894	1.01801	1.01875	1.01782	1.00015	1.00013	
Set-1	Day	100	1.00114	1.00049	1.00066	1.00008	1.00906	1.00858	1.00878	1.00830	1.00012	1.00011	
Set-1	Week	100	1.00115	1.00056	1.00202	1.00173	1.01113	1.01053	1.01108	1.01048	1.00012	1.00011	
Set-1	Month	100	1.01472	1.01332	1.01115	1.01096	1.01256	1.01248	1.01166	1.01157	1.00015	1.00012	
Set-2	Day	30	1.00012	0.99970	1.01171	1.01075	1.00213	1.00192	1.00166	1.00145	1.00012	0.99999	
Set-2	Week	30	1.00923	1.00600	1.01214	1.01018	1.00929	1.00737	1.00969	1.00697	1.00011	1.00011	
Set-2	Month	30	1.01238	1.01020	1.02036	1.01960	1.01206	1.01101	1.01158	1.01053	1.00011	0.99999	
Set-2	Day	50	1.00016	0.99962	1.00994	1.00909	1.00125	1.00102	1.00095	1.00072	1.00011	1.00011	
Set-2	Week	50	1.00671	1.00337	1.01670	1.01535	1.00639	1.00477	1.00725	1.00391	1.00011	1.00011	
Set-2	Month	50	1.01235	1.00744	1.02279	1.02271	1.01203	1.00945	1.01134	1.00876	1.00011	1.00011	
Set-2	Day	100	1.00010	0.99970	1.00916	1.00795	1.00130	1.00106	1.00057	1.00033	1.00002	1.00002	
Set-2	Week	100	1.00940	1.00613	1.01684	1.01632	1.00934	1.00754	1.00839	1.00659	1.00012	1.00012	
Set-2	Month	100	1.01487	1.01233	1.02385	1.02317	1.01516	1.01354	1.01508	1.01332	1.00012	1.00012	
Set-3	Day	30	0.99999	0.99980	1.00116	1.00040	1.00190	1.00176	1.00120	1.00105	1.00010	1.00004	
Set-3	Week	30	1.01331	1.00910	1.02303	1.02294	1.01333	1.01277	1.01287	1.01231	1.00003	0.99999	
Set-3	Month	30	1.00143	1.00132	1.03006	1.02983	1.01543	1.01419	1.01465	1.01341	1.00010	1.00009	
Set-3	Day	50	1.00115	1.00070	1.00582	1.00511	1.00114	0.99979	1.00183	1.00048	1.00010	1.00009	
Set-3	Week	50	0.99993	0.99992	1.02561	1.02518	1.02310	1.02171	1.02258	1.02119	1.00004	0.99999	
Set-3	Month	50	1.00096	1.00051	1.03666	1.03568	1.02910	1.02875	1.02876	1.02841	1.00002	0.99999	
Set-3	Day	100	1.00052	1.00034	1.01907	1.01873	1.00252	1.00053	1.00343	1.00145	1.00011	1.00009	
Set-3	Week	100	1.01460	1.00999	1.02776	1.02722	1.02436	1.02398	1.02383	1.02346	1.00010	1.00009	
Set-3	Month	100	1.00184	1.00173	1.03989	1.03920	1.02847	1.02758	1.02821	1.02731	1.00010	1.00009	
Set-4	Day	30	1.00151	1.00139	1.00048	1.00045	1.00150	1.00132	1.00184	1.00166	1.00010	1.00009	
Set-4	Week	30	1.00126	1.00091	1.00343	1.00324	1.00126	1.00085	1.00109	1.00068	1.00010	1.00009	
Set-4	Month	30	1.00265	1.00243	1.01224	1.01207	1.00256	1.00242	1.00188	1.00174	1.00010	1.00009	
Set-4	Day	50	1.00066	1.00041	1.00165	1.00156	1.00066	1.00040	1.00092	1.00066	1.00010	1.00008	
Set-4	Week	50	1.03039	1.02715	1.00888	1.00871	1.03024	1.02674	1.02834	1.02485	1.00000	0.99998	
Set-4	Month	50	0.99819	0.99551	1.01386	1.01379	1.02013	1.01780	1.01947	1.01714	1.00010	1.00008	
Set-4	Day	100	1.01098	1.00777	1.00269	1.00257	1.01098	1.00749	1.01064	1.00715	1.00010	1.00009	
Set-4	Week	100	1.00630	1.00426	1.01174	1.01168	1.00626	1.00386	1.00571	1.00331	1.00010	1.00009	
Set-4	Month	100	1.02177	1.01837	1.01976	1.01971	1.02079	1.01736	1.02081	1.01738	1.00010	1.00008	

4.1.4. Closeness to Optimality

In this section, we measure the closeness of the solutions proposed by the Linear Approximation method to optimality. We assume that the optimal solution is obtained by the Normal (Log-Normal) Approximation method when the data set is composed of Normally (Log-Normally) distributed stock return scenarios since the Normal (Log-Normal) Approximation method assumes that the stock returns are Normally (Log-Normally) distributed.

Multi-variate Normally and Log-Normally distributed rate of return (ROR) scenarios are generated according to the sample mean and standard deviation of historical data set composed of 100 observations of 30 stocks listed in NYSE. Sixty different cases are considered, namely, the cases where the data sets are composed of 500, 1000 and 2000 scenarios for Normally and Log-Normally distributed RORs of 10, 20, 30,..,90, and 100 assets. Figure 1, Figure 2, and Figure 3 represent the empirical probability distribution of averages of Normally and Log-Normally distributed 1000 ROR scenarios of 10, 50 and 100 stocks, respectively.

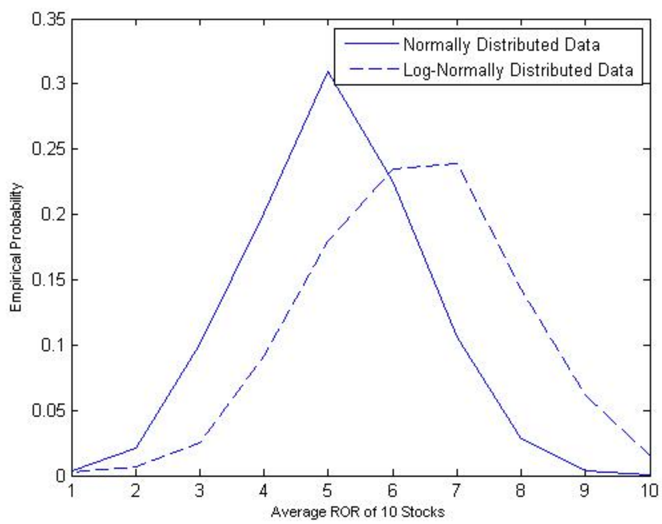

Figure 1: Empirical Distribution of Average ROR of 10 Stocks

We define the measure for closeness to optimality, θ, as the relative difference between the objective function values of the Linear Approximation and Normal (LogNormal) Approximation methods, where

$$\theta = \frac{|Obj^* - Obj_{App}|}{|Obj^*|}.$$

Table 3 and Table 4 provide the 95^{th} quantile value for θ obtained by comparing the proposed Linear Approximation method and the benchmark approximation methods (the Normal approximation and the log-Normal approximation)

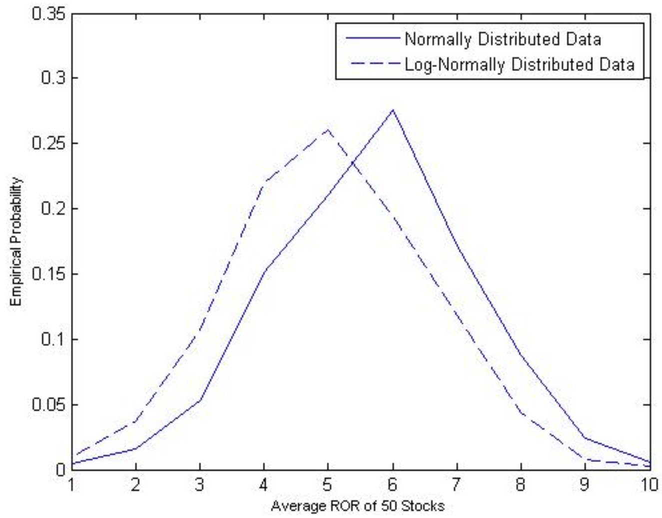

Figure 2: Empirical Distribution of Average ROR of 50 Stocks

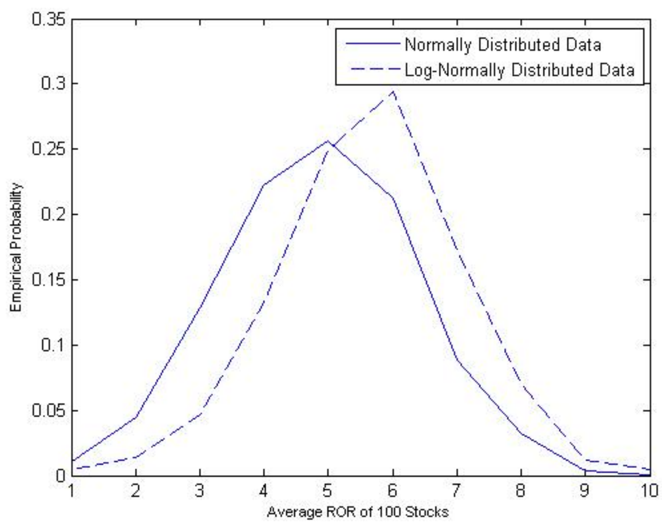

Figure 3: Empirical Distribution of Average ROR of 100 Stocks

over samples of observations. For each data set, the Linear Approximation method and the corresponding benchmark model are run multiple times with different quantile targets (q_m), then θ values are obtained for each single run. Next, the 95^{th} quantile value for θ is calculated from the sample of θ specific to the corresponding data set. Table 3 and Table 4 suggest that solutions suggested provided by the Linear Approximation approach are close to optimality.

We use also the Brute Force method as another benchmark to measure the closeness of the Linear Approximation method to optimality. The Brute Force method consists in enumerating all possible candidates for the solution and selecting the one which satisfies the constraints and provides the best objective function. Therefore, it leads to the optimal solution (Rodriguez [19]).

In this study, 2 assets and 500 observations of both are considered. Both the Linear Approximation method and the Brute Force method are run for the same q_m targets and the 95^{th} quantile value is calculated for 10 different data sets. The results are summarized in Table 5.

Table 3: 95^{th} Quantile for θ, Normally Distributed Data

Number of Scenarios	Number of Assets	95^{th} Quantile for θ
500	10	0.000101
500	20	0.000017
500	30	0.000302
500	40	0.000333
500	50	0.000103
500	60	0.000272
500	70	0.000178
500	80	0.000112
500	90	0.000133
500	100	0.000099
1000	10	0.000089
1000	20	0.000156
1000	30	0.000083
1000	40	0.000193
1000	50	0.000188
1000	60	0.000385
1000	70	0.000371
1000	80	0.000032
1000	90	0.000107
1000	100	0.000061
2000	10	0.000033
2000	20	0.000077
2000	30	0.000115
2000	40	0.000343
2000	50	0.000071
2000	60	0.000063
2000	70	0.000102
2000	80	0.000143
2000	90	0.000126
2000	100	0.000184

4.2. Portfolio management with interquartile range minimization

In this section, we present numerical results pertaining to CPU time and number of iterations for IQR minimization. Each row in Table 6 represents a set of experiments where the interquartile range management problem is solved with several expected portfolio return targets over the data set with the same number of observations and assets. The numerical experiments are repeated with different data sets having various number of scenarios and assets.

From Table 6, we see that the number of iterations and solution time for the inter-quartile range problem heuristic are slightly higher than those for the quantile management algorithm presented in the previous section. As the number of scenarios increases, the number of iterations also increases, because determining the worst case scenarios leading to the min inter-quartile range value becomes harder as more scenarios are considered. Additional results, including efficient frontier graphs, are provided in Çetinkaya [38].

5. Conclusions

In this paper, we investigated an approximation method to solve the portfolio management problem with quantile constraints, with an extension to the interquartile range minimization problem. The algorithm involves solving a series of linear problems iteratively and is thus highly tractable. Our numerical experiments suggest that our method leads to high-quality portfolio allocation decisions.

Appendix 1

Denote $e^{R_i^t}$ the return of stock i during time period t. Then return of stock i from time 1 to time T is $e^{\sum_{t=1}^{T} R_i^t}$. Therefore, the portfolio return over T period can be formulated as:

$$W = \sum_{i=1}^{n} x_i e^{\sum_{t=1}^{T} R_i^t}.$$

Then, the first and the second moments of the portfolio return are calculated as:

$$E[W] = \sum_{i=1}^{T} x_i E[e^{\sum_{t=1}^{T} R_i^t}] = \sum_{i=1}^{n} e^{(\tilde{\mu_i} T + \frac{\tilde{\sigma_i}^2 T}{2})} \tag{11}$$

$$\begin{aligned} E[W^2] &= E\left[\left(\sum_{i=1}^{T} x_i e^{\sum_{t=1}^{T} R_i^t}\right)^2\right] \\ &= \sum_{i=1}^{n}\left(x_i^2 e^{2T\tilde{\mu_i}+2T\tilde{\sigma_i}^2} + \sum_{j=1,j\neq i}^{n} x_i x_j e^{\left((\tilde{\mu_i}+\tilde{\mu_j})T+\frac{T}{2}(\tilde{\sigma_i}^2+\tilde{\sigma_j}^2+2\rho_{i,j}\tilde{\sigma_i}\tilde{\sigma_j})\right)}\right) \end{aligned} \tag{12}$$

We define the vector $b \in \mathcal{R}^n$ such that

$$b_i = e^{(\tilde{\mu_i} T + \frac{\tilde{\sigma_i}^2 T}{2})} \ \forall i,$$

and the matrix A $\in \mathcal{R}^{nxn}$ such that

$$A_{i,j} = e^{\left((\tilde{\mu_i} + \tilde{\mu_j})T + \frac{T}{2}(\tilde{\sigma_i}^2 + \tilde{\sigma_j}^2 + 2\rho_{i,j}\tilde{\sigma_i}\tilde{\sigma_j})\right)} \ \forall i, \ \forall j, \text{ and } i \neq j$$

$$A_{i,i} = e^{2T\tilde{\mu_i} + 2T\tilde{\sigma_i}^2} \ \forall i.$$

The Log-Normal approximation of the portfolio return is represented as e^Y where $Y \sim N(\mu^*, \sigma^*)$. Then the following equations hold:

$$\begin{aligned} E[W] &= b'x = E[e^Y] = e^{\mu^* + \frac{\sigma^{*2}}{2}} \\ E[W^2] &= x'Ax = e^{2\mu^* + 2\sigma^{*2}} \end{aligned} \tag{13}$$

The solution of this system of equations is as follows:

$$\begin{aligned} \mu^* &= 2ln(b'x) - \frac{1}{2}ln(x'Ax) \\ \sigma^{*2} &= ln(x'Ax) - 2ln(b'x) \end{aligned} \tag{14}$$

Then, the expected return maximization problem with quantile constraint is written as:

$$\begin{aligned} \max \quad & \left(e^{\mu^* + \frac{\sigma^{*2}}{2}}\right) \\ \text{s.t.} \quad & \mu^* + \phi^{-1}(\alpha)\sigma^* \geq ln(q_m), \\ & x \in X, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \max \quad & b^T x \\ \text{s.t.} \quad & 2\ln(b^T x) - \frac{1}{2}\ln(x^T A x) + \phi^{-1}(\alpha)\sqrt{\ln(b^T x) - 2\ln(x^T A x)} \geq \ln(q_m), \\ & x \in X. \end{aligned} \tag{15}$$

References

[1] J. Keisler, Value of information in portfolio decision analysis, Decision Analysis 1 (3) (2004) 177–189.

[2] R. T. Rockafellar, S. Uryasev, Optimization of conditional value at risk, Journal of Risk 2 (3) (2000) 21–41.

[3] W. V. Harlow, Asset allocation in a downside risk framework, Financial Analysts Journal 47 (5) (1991) 28–40.

[4] G. Cornuejols, R. Tutuncu, Optimization Methods in Finance, Cambridge University Press, New York, NY, 2007.

[5] H. M. Markovitz, Portfolio selection, Journal of Finance 7 (1) (1952) 77–91.

[6] H. M. Markovitz, Portfolio Selection, John Wiley and Sons, New York, NY, 1959.

[7] A. Roy, Safety first and the holding of assets, Econometrica 20 (3) (1952) 431–449.

[8] W. F. Sharpe, Mutual fund performance, The Journal of Business 39 (1966) 119–138.

[9] F. A. Sortino, L. N. Price, Performance measurement in downside risk framework, Journal of Investing 3 (1994) 59–64.

[10] G. Szegö, Measures of risk, Journal of Banking & Finance 26 (2002) 1253–1272.

[11] W. F. Sharpe, Mean absolute deviation characteristic lines for securities and portfolios, Management Science 18 (2) (1971) B1–B13.

[12] S. Yitzhaki, Stochastic dominance, mean variance, and gini's mean difference, American Economic Association 72 (1) (1982) 178–185.

[13] P. Artzner, F. Delbaen, J.-M. Eber, D. Health, Coherent measures of risk, Mathematical Finance 9 (3) (1999) 203–228.

[14] D. Dentcheva, A. Ruszczynski, Portfolio optimization with stochastic dominance constraints, Journal of Banking and Finance 30 (2) (2006) 433–451.

[15] A. Ruszczynski, R. Vanderbei, Frontiers of stochastically nondominated portfolios, Econometrica 71 (4) (2003) 1287–1297.

[16] S. Benninga, Z. Wiener, Value-at-risk (var), Mathematica in Education and Research 7 (4).

[17] T. J. Linsmeier, N. D. Pearson, Value at risk, Financial Analyst Journal 56 (2) (2000) 47–67.

[18] J. H. Kim, W. B. Powell, Quantile optimization for heavy-tailed distribution using asymmetric signum functions, Princeton University.

[19] G. J. L. Rodriguez, Portfolio optimization with quantile-based risk measures, Ph.D. thesis, Massachusetts Institute of Technology, Massachusetts, USA (1999).

[20] S. Uryasev (Ed.), Probabilistic Constrained Optimization: Methodology and Applications, Springer, New York, NY, 2000.

[21] A. V. Naumov, A. I. Kibzun, Quantile optimization techniques with application to chance constrained problem for water-supply system design, Tech. Rep. 92-5, Department of Industrial and Operations Engineering at University of Michigan and Department of Applied Mathematics Moscow Aviation Institute, Ann Arbor. Mi 48109 and Moskiw, 127080, Russia (January 1992).

[22] A. R. Pankov, E. N. Platonov, K. V. Semenikhin, Minimax optimization of investment portfolio by quantile criterion, Automation and Remote control 64 (7) (2002) 1122–1137.

[23] L. El-Ghaoui, L. Oks, F. Oustry, Worst-case value-at-risk and robust asset allocation: A semidefinite programming approach, Tech. Rep. M00/59, University of California, Berkeley (January 2000).

[24] S. Benati, R. Rizzi, A mixed integer linear programming formulation of the optimal mean/value-at-risk portfolio problem, European Journal of Operational Research 176 (2007) 423–434.

[25] A. Gaivoronski, G. Pflug, Value-at-risk in portfolio optimization: properties and computational approach, Journal of Risk 7 (2) (2005) 1–31.

[26] F. Fabozzi, P. Kolm, D. Pachamanova, S. Focardi, Robust portfolio optimization and management, Wiley, New York, NY, 2007.

[27] D. Wozabal, Value-at-risk optimization using the difference of convex algorithm, OR Spectrum 34 (2012) 681–683.

[28] J. Goh, K. Lim, M. Sim, W. Zhang, Portfolio value-at-risk optimization for asymmetrically distributed asset returns, European Journal of Operational Research 221 (2) (2012) 397–406.

[29] S. Zymler, D. Kuhn, B. Rustem, Worst-case value-at-risk of nonlinear portfolios, Management Science 59 (1) (2013) 172–188.

[30] S. Rachev, S. Stoyanov, F. Fabozzi, Advanced stochastic models, risk assessment and portfolio optimization: the ideal risk, uncertainty and performance measures, Wiley, New York, NY, 2008.

[31] J. Beasley, Portfolio optimization: models and solution approaches, Tutorials in Operations Research 1 (2013) 201–221.

[32] M. Colombo, Advances in interior point methods for large scale linear programming, Ph.D. thesis, Doctor of Philosophy University of Edinburgh (2007).

[33] M. Lobo, M. Fazel, S. Boyd, Portfolio optimization with linear and fixed transaction costs, Ann Oper Res 152 (5).

[34] N. Larsen, H. Mausser, S. G., S. Urysasev, Algorithms for Optimization of Value-at-Risk. In Pardalos P. and Tsitsiringos, V. K., editors, Vol. 126 of Applied Optimization Series, Kluwer Academic Publishers, Boston, 2002.

[35] L. F. Fenton, The sum of lognormal probability distributions in scatter transmission systems, IRE Trans. Commun. Syst. CS-8 (3) (1969) 57–67.

[36] B. Pfaff, Financial risk modeling and portfolio optimization with R, Wiley, New York, NY, 2013.

[37] T. Oyama, Determinants of stock prices: The case of zimbabwe, a Working Paper of the International Monetary Fund (2007).

[38] E. Cetinkaya, Essays in robust and data-driven risk management, Ph.D. thesis, Lehigh University, Bethlehem, PA, USA (2014).

Table 4: 95^{th} Quantile for θ, Log-Normally Distributed Data

Number of Scenarios	Number of Assets	95^{th} Quantile for θ
500	10	0.000175
500	20	0.000403
500	30	0.000136
500	40	0.000426
500	50	0.000269
500	60	0.000279
500	70	0.000214
500	80	0.000220
500	90	0.000212
500	100	0.000131
1000	10	0.000049
1000	20	0.000003
1000	30	0.000148
1000	40	0.001165
1000	50	0.000088
1000	60	0.000216
1000	70	0.000282
1000	80	0.000074
1000	90	0.000528
1000	100	0.000529
2000	10	0.000116
2000	20	0.000158
2000	30	0.000050
2000	40	0.000110
2000	50	0.000180
2000	60	0.000136
2000	70	0.000140
2000	80	0.000052
2000	90	0.000056
2000	100	0.000118

Table 5: 95^{th} Quantile for θ, Benchmark: Brute Force Method

Data Set Index	95% CI for θ
1	0.000383
2	0.000045
3	0.000104
4	0.000181
5	0.000025
6	0.000002
7	0.000047
8	0.000032
9	0.000063
10	0.000006

Table 6: Amount of Time (CPU Seconds) and Number of Iterations to Convergence

Sample Size	Asset Number	Iteration Range	Min. Solution Time	Max. Solution Time
100	30	[2,6]	0.0600	0.3240
1000	30	[3,7]	0.3120	0.8401
2000	30	[2,18]	0.3200	6.4924
5000	30	[2,16]	1.0121	7.6165
100	50	[3,4]	0.1720	0.2440
1000	50	[2,13]	0.3120	4.4723
2000	50	[5,16]	3.1002	10.4726
5000	50	[4,17]	7.6085	24.4694
100	100	[5,7]	1.0521	1.2841
1000	100	[5,10]	3.4723	8.8246
2000	100	[4,12]	7.6565	24.1252
5000	100	[3,6]	10.1850	26.4140
100	200	[2,5]	0.5120	1.9081
1000	200	[3,12]	31.3380	133.0380
2000	200	[2,15]	9.9006	63.2880
5000	200	[3,18]	40.0430	293.5320